



TITLE:

# A new proof of Carlson's theorem by Plana's summation formula (Asymptotic Analysis and Microlocal Analysis of PDE)

AUTHOR(S):

Yoshino, Kunio; Suwa, Masanori

---

CITATION:

Yoshino, Kunio ...[et al]. A new proof of Carlson's theorem by Plana's summation formula (Asymptotic Analysis and Microlocal Analysis of PDE). 数理解析研究所講究録 2001, 1211: 166-174

ISSUE DATE:

2001-06

URL:

<http://hdl.handle.net/2433/41136>

RIGHT:

# A new proof of Carlson's theorem by Plana's summation formula

吉野 邦生 (Kunio Yoshino)  
諏訪 将範 (Masanori Suwa)

Department of Mathematics, Sophia University

## Abstract

In this paper we will prove Carlson's theorem by using Plana's summation formula.

In §1, we will recall the definitions and transformations of analytic functionals with unbounded carrier.

In §2, we will give a proof of Carlson's theorem by Plana's summation formula.

## 1 The definitions of analytic functionals with unbounded carrier and their transformations.

Let  $L$  and  $L_\varepsilon$  be following strip regions:

$$\begin{aligned} L &= (-\infty, a] + i[-b, b], \\ \text{For } \varepsilon > 0, \quad L_\varepsilon &= (-\infty, a + \varepsilon] + i[-b - \varepsilon, b + \varepsilon]. \end{aligned}$$

For  $\varepsilon > 0$ ,  $\varepsilon' > 0$  and  $k' \in \mathbb{R}$ , we put

$$\begin{aligned} &Q_b(L_\varepsilon : k' + \varepsilon') \\ &:= \left\{ f(\zeta) \in \mathcal{H}(\overset{\circ}{L}_\varepsilon) \cap \mathcal{C}(L_\varepsilon) : \sup_{\zeta \in L_\varepsilon} |f(\zeta)| e^{k'\xi + \varepsilon'|\xi|} < \infty, \quad \zeta = \xi + i\eta \right\}. \end{aligned}$$

$\mathcal{H}(\check{L}_\varepsilon)$  is the space of holomorphic functions defined on  $\check{L}_\varepsilon$ , (interior of  $L_\varepsilon$ ).  $\mathcal{C}(L_\varepsilon)$  is the space of continuous functions defined on  $L_\varepsilon$ . We put

$$Q(L : k') = \varinjlim_{\varepsilon \rightarrow 0, \varepsilon' \rightarrow 0} Q_b(L_\varepsilon : k' + \varepsilon'),$$

where  $\varinjlim$  means inductive limit. If  $z \in (-k', \infty) + i\mathbb{R}$ , then the function  $e^{\zeta z}$  of  $\zeta$  belongs to  $Q(L : k')$ . We denote by  $Q'(L : k')$  the dual space of  $Q(L : k')$ . The elements of  $Q'(L : k')$  is called analytic functionals with carrier  $L$  and of type  $k'$ .

We define the Fourier-Borel transform  $\tilde{T}(z)$  of  $T \in Q'(L : k')$  as follows:

$$\tilde{T}(z) = \langle T_\zeta, e^{\zeta z} \rangle.$$

$\tilde{T}(z)$  is holomorphic function on the right half plane  $(-k', \infty) + i\mathbb{R}$  and satisfies following estimate :

$$\forall \varepsilon > 0, \varepsilon' > 0, \exists C_{\varepsilon, \varepsilon'} \geq 0 \text{ such that}$$

$$|\tilde{T}(z)| \leq C_{\varepsilon, \varepsilon'} e^{ax+b|y|+\varepsilon|z|}, \quad (\operatorname{Re} z \geq -k' + \varepsilon', \quad z = x + iy). \quad (1)$$

$\operatorname{Exp}((-k', \infty) + i\mathbb{R} : L)$  denotes the space of holomorphic functions defined on the right half plane  $(-k', \infty) + i\mathbb{R}$  and satisfy the estimates (1). Following theorem characterizes the Fourier-Borel transform of  $Q'(L : k')$ .

**Theorem 1.1** ([2],[9]). *Fourier-Borel transform is a linear topological isomorphism from  $Q'(L : k')$  onto  $\operatorname{Exp}((-k', \infty) + i\mathbb{R} : L)$ .*

**Definition 1.2.**  $\varepsilon'$ -Cauchy transform  $\tilde{T}(w, \varepsilon')$  of  $T \in Q'(L : k')$  is defined as follows :

$$\tilde{T}(w, \varepsilon') = \frac{-1}{2\pi i} \left\langle T_\zeta, \frac{e^{(-k' + \varepsilon')(\zeta - w)}}{\zeta - w} \right\rangle.$$

**Proposition 1.3** ([2],[9]). *The following integral representation holds :*

$$\langle T, \varphi \rangle = \frac{1}{2\pi i} \int_{\partial L_\varepsilon} \tilde{T}(\zeta, \varepsilon') \varphi(\zeta) d\zeta, \quad (\forall \varphi \in Q_b(L_\varepsilon : k' + \varepsilon'_1), \varepsilon' < \varepsilon'_1).$$

Put  $\varphi(\zeta) = e^{\zeta z}$ , then we have

$$\tilde{T}(z) = \frac{1}{2\pi i} \int_{\partial L_\varepsilon} \tilde{T}(\zeta, \varepsilon') e^{\zeta z} d\zeta.$$

Conversely, we can express  $\tilde{T}$  by  $\tilde{T}$  in as follows :

$$\tilde{T}(w, \varepsilon') = \int_{-k'+\varepsilon'}^{\infty} \tilde{T}(z) e^{-wz} dz.$$

**Definition 1.4.** For  $S \in \mathcal{H}'(K)$ ,  $K$  is a compact set and  $T \in Q'(L : k')$ , we define convolution  $*$  by

$$T * S = S * T = \langle T_{\zeta}, \langle S_{\tau}, \varphi(\tau + \zeta) \rangle \rangle, \quad \varphi \in Q'(L + K : k').$$

**Proposition 1.5.** We have the following equalities :

$$(i) \quad T * S \in Q'(L + K : k'),$$

$$(ii) \quad \widetilde{(T * S)}(z) = \tilde{T}(z) * \tilde{S}(z).$$

In [15] we derived Plana's summation formula for holomorphic functions of exponential type by using the theory of analytic functionals with unbounded carrier.

**Proposition 1.6 (Plana's summation formula [6], [13],[15]).** Let  $T \in Q'(L : k')$ . If  $k' > 0$ ,  $0 \leq b < 2\pi$ ,  $\text{Res} > a$  and  $|\text{Im}s| + b < 2\pi$ , then we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \tilde{T}(n) e^{-sn} \\ &= \frac{1}{2} \tilde{T}(0) + \int_0^{\infty} \tilde{T}(x) e^{-sx} dx + i \int_0^{\infty} \frac{\tilde{T}(ix) e^{-isx} - \tilde{T}(-ix) e^{isx}}{e^{2\pi x} - 1} dx. \end{aligned}$$

## 2 Applications.

In this section we determine the form of holomorphic functions  $f(z)$  of exponential type with  $f(\mathbb{N}) = \{0\}$ .

**Lemma 2.1.** Suppose that  $T \in Q'(L : k')$ ,  $L = (-\infty, a] + i[-b, b]$ ,  $0 < k' < 1$ ,  $0 \leq b < 2\pi$ . If  $\tilde{T}(n) = 0$ , ( $n = 0, 1, 2, \dots$ ), then there exist holomorphic functions  $a(s)$  and  $b(s)$  such that

$$\tilde{T}(s, \varepsilon') = a(s) + b(s),$$

where  $a(s) \in \mathcal{H}(\{s \in \mathbb{C} : \text{Im}s < 2\pi - b\})$ ,  $b(s) \in \mathcal{H}(\{s \in \mathbb{C} : \text{Im}s > b - 2\pi\})$  and satisfy the following estimate :

$$\begin{aligned} |a(s)| &\leq C_{\varepsilon'} e^{(k' - \varepsilon') \text{Re}s}, \\ |b(s)| &\leq C_{\varepsilon'} e^{(k' - \varepsilon') \text{Re}s}. \end{aligned}$$

**Corollary 2.2.** Let  $T \in Q'(L : k')$ . If  $\tilde{T}(n) = 0$  ( $n = 0, 1, 2, \dots$ ), then  $\tilde{T}(s, \varepsilon')$  is holomorphic in  $|\operatorname{Im} s| < 2\pi - b$ , and  $|\tilde{T}(s, \varepsilon')| \leq C_{\varepsilon'} e^{(k' - \varepsilon') \operatorname{Re} s}$ .

**Proof of lemma 2.1.** By Plana's summation formula and assumption  $\tilde{T}(n) = 0$ , ( $n = 0, 1, 2, \dots$ ), we have

$$\begin{aligned}
0 &= \int_0^\infty \tilde{T}(x) e^{-xs} dx + i \int_0^\infty \frac{\tilde{T}(ix) e^{-ixs} - \tilde{T}(-ix) e^{ixs}}{e^{2\pi x} - 1} dx \\
&= \int_0^\infty \tilde{T}(x) e^{-xs} dx + i \int_0^\infty \frac{\tilde{T}(ix) e^{-ixs}}{e^{2\pi x} - 1} dx - i \int_0^\infty \frac{\tilde{T}(-ix) e^{ixs}}{e^{2\pi x} - 1} dx \\
&= \int_0^\infty \tilde{T}(x) e^{-xs} dx + \int_0^{i\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi iz} - 1} dz \\
&\quad - \int_0^{-i\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi iz} - 1} dz - \int_0^{-i\infty} \tilde{T}(z) e^{-sz} dz \\
&= \int_0^\infty \tilde{T}(x) e^{-xs} dx + \int_{-k' + \varepsilon'}^0 \tilde{T}(x) e^{-xs} dx + \int_0^{i\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi iz} - 1} dz \\
&\quad - \int_0^{-i\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi iz} - 1} dz - \int_0^{-i\infty} \tilde{T}(z) e^{-sz} dz - \int_{-k' + \varepsilon'}^0 \tilde{T}(x) e^{-xs} dx \\
&= \int_{-k' + \varepsilon'}^\infty \tilde{T}(x) e^{-xs} dx + \int_{-k' + \varepsilon'}^{i\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi iz} - 1} dz \\
&\quad - \int_{-k' + \varepsilon'}^{-i\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi iz} - 1} dz - \int_{-k' + \varepsilon'}^{-i\infty} \tilde{T}(z) e^{-sz} dz.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{-k' + \varepsilon'}^\infty \tilde{T}(x) e^{-xs} dx \\
&= - \int_{-k' + \varepsilon'}^{i\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi iz} - 1} dz + \int_{-k' + \varepsilon'}^{-i\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi iz} - 1} dz + \int_{-k' + \varepsilon'}^{-i\infty} \tilde{T}(z) e^{-sz} dz.
\end{aligned}$$

Put

$$\begin{aligned}
a(s) &= - \int_{-k' + \varepsilon'}^{i\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi iz} - 1} dz, \\
b(s) &= \int_{-k' + \varepsilon'}^{-i\infty} \frac{\tilde{T}(z) e^{-sz}}{e^{-2\pi iz} - 1} dz + \int_{-k' + \varepsilon'}^{-i\infty} \tilde{T}(z) e^{-sz} dz.
\end{aligned}$$

Since by proposition 1.3,  $\tilde{T}(s, \varepsilon') = \int_{-k'+\varepsilon'}^{\infty} \tilde{T}(z)e^{-sz}dz$ , we have  $\tilde{T}(s, \varepsilon') = a(s) + b(s)$ . Then  $a(s)$  and  $b(s)$  satisfy the conditions in lemma 2.1. ■

**Proposition 2.3.** *Suppose that  $T \in Q'(L : k')$ ,  $L = (-\infty, a] + i[-b, b] \subset \{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < 2\pi\}$ ,  $0 < k' < 1$ .  $\tilde{T}(n) = 0$ , ( $n = 0, 1, 2, \dots$ ).*

(i) *If  $0 \leq b < \pi$ , then  $T \equiv 0$ .*

(ii) *If  $\pi \leq b < 2\pi$ , then there exists an analytic functional  $S$  with unbounded carrier such that  $T = (\delta_{i\pi} - \delta_{-i\pi}) * S$ ,  $S \in Q'((-\infty, a] + i[\pi - b, b - \pi] : k')$ , where  $*$  is convolution.*

**Proof.** (i)

$$\langle T, \varphi \rangle = \frac{1}{2\pi i} \int_{\partial L_\varepsilon} \tilde{T}(\zeta, \varepsilon') \varphi(\zeta) d\zeta, \quad (\varphi \in Q_b(L_\varepsilon : k' + \varepsilon'_1), \quad \varepsilon' < \varepsilon'_1),$$

and  $\tilde{T}(\zeta, \varepsilon') \in \mathcal{H}(\mathbb{C} \setminus L)$ . By lemma 2.1,  $\tilde{T}(\zeta, \varepsilon') \in \mathcal{H}(\{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < 2\pi - b\})$ . From the assumption  $0 \leq b < \pi$ ,  $L$  is contained in  $\{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < 2\pi - b\}$  and  $|\tilde{T}(\zeta, \varepsilon')| \leq Ce^{(k'-\varepsilon')\zeta}$ . Hence  $\langle T, \varphi \rangle = 0$ .

(ii) In this case  $\tilde{T}(\zeta, \varepsilon') \in \mathcal{H}(\mathbb{C} \setminus \{(-\infty, a] + i[2\pi - b, b] \cup (-\infty, a] + i[-b, b - 2\pi]\})$ . We define analytic functionals with unbounded carrier  $S_1, S_2$  as follows :

$$\begin{aligned} \langle S_1, \varphi \rangle &= \frac{1}{2\pi i} \int_{\partial L'_\varepsilon} \tilde{T}(\zeta + \pi i) \varphi(\zeta) d\zeta, \\ \langle S_2, \varphi \rangle &= \frac{1}{2\pi i} \int_{\partial L'_\varepsilon} \tilde{T}(\zeta - \pi i) \varphi(\zeta) d\zeta, \end{aligned}$$

where  $\varphi \in Q_b((-\infty, a] + i[\pi - b, b - \pi] : k' + \varepsilon')$ ,  $L'_\varepsilon = (-\infty, a + \varepsilon] + i[\pi - b - \varepsilon, b - \pi + \varepsilon]$ . By lemma 2.1,  $S_1, S_2 \in Q'((-\infty, a] + i[\pi - b, b - \pi] : k')$  and we have  $T = S_1 * \delta_{\pi i} + S_2 * \delta_{-\pi i}$ . From the assumption  $\tilde{T}(n) = 0$ , ( $n = 0, 1, 2, \dots$ ),

$$\begin{aligned} 0 &= \tilde{T}(n) \\ &= \tilde{S}_1(n)e^{n\pi i} + \tilde{S}_2(n)e^{-n\pi i}. \end{aligned}$$

Therefore  $\tilde{S}_1(n) = -\tilde{S}_2(n)$ . By  $b - \pi - (\pi - b) = 2(b - \pi) < 2\pi$  and (i) in this proposition,  $S_1 = -S_2$ . ■

**Corollary 2.4.** Let  $f(z) \in \text{Exp}((-k', \infty) + i\mathbb{R} : L)$ ,  $0 < k' < 1$ ,  $f(n) = 0$ , ( $n = 0, 1, 2, \dots$ ). Then

$$(i) \quad 0 \leq b < \pi \implies f(z) = 0.$$

$$(ii) \quad \pi \leq b < 2\pi \implies f(z) = \sin \pi z \tilde{S}(z), \quad \text{where } S \in Q'((-\infty, a] + i[\pi - b, b - \pi] : k').$$

**Remark 2.5.** Corollary 2.4(i) are called “Carlson’s theorem”. ([3])

**Corollary 2.6.**

$$(i) \quad 0 \leq b < \pi \implies \{e^{n\zeta}\}_{n=0}^{\infty} \text{ is total in } Q(L : k').$$

$$(ii) \quad \pi \leq b < 2\pi \implies \{e^{m\zeta}, \zeta e^{n\zeta}\}_{m=0}^{\infty} \text{ is total in } Q(L : k').$$

## 2.1 Examples

Carlson’s theorem is used in several branches of mathematical physics. For example,

1. A uniqueness of analytic continuation of scattering amplitude to complex angular momentum plane ([7], [11],[12]).
2. Dyson Conjecture in statistical mechanics ([5], [8]).
3. Calculation of Selberg Integral ([1],[8]).
4. A uniqueness of Ramanujan Resummation ([4]).

For the history of Carlson’s theorem, we refer the reader to [14]. Here we have an example of proposition 2.3 (ii) :

**Example 2.7 (Functional equation of Riemann zeta function, [6],[13]).**

We define an analytic functional  $T$  as follows :

$$\langle T, \varphi \rangle = \frac{1}{2\pi i} \int_{\partial L_\epsilon} \frac{1}{2(e^{-e^{-\frac{1}{2}\zeta}} - 1)} \varphi(\zeta) d\zeta, \quad \varphi \in Q(L : k'), \quad (2)$$

where  $L_\varepsilon = (-\infty, -2\log 2\pi + \varepsilon] + i[-\pi - \varepsilon, \pi + \varepsilon]$ ,  $\varepsilon > 0$ . In (2), we put  $t = e^{-\frac{1}{2}\zeta}$  and  $\varphi(\zeta) = e^{\zeta z}$ . Then we have

$$\begin{aligned}\tilde{T}(z) &= \langle T_\zeta, e^{\zeta z} \rangle \\ &= -\frac{1}{2\pi i} \int_{(+\infty)}^{(+0)} \frac{(-t)^{-2z-1}}{e^t - 1} dt \\ &= \frac{\zeta(-2z)}{\Gamma(2z+1)}.\end{aligned}\tag{3}$$

Remark that  $\tilde{T}(n) = \frac{\zeta(-2n)}{\Gamma(2n+1)} = 0$ . On the other hand, we have

$$\left| \frac{1}{2\pi i} \int_{\partial L_\varepsilon} \frac{e^{\zeta z}}{2(e^{-e^{-\frac{1}{2}\zeta}} - 1)} d\zeta \right| \rightarrow 0, \quad (\varepsilon \rightarrow -\infty, \quad \operatorname{Re} z > 0).$$

Therefore by residue theorem,

$$\begin{aligned}\frac{1}{2\pi i} \int_{\partial L_\varepsilon} \frac{e^{\zeta z}}{2(e^{-e^{-\frac{1}{2}\zeta}} - 1)} d\zeta &= -\sum_{n=0}^{\infty} \frac{(2\pi n)^{-2z} e^{\pi i z}}{2\pi i n} + \sum_{n=0}^{\infty} \frac{(2\pi n)^{-2z} e^{-\pi i z}}{2\pi i n} \\ &= -\frac{(2\pi)^{-2z}}{\pi} \sin \pi z \zeta(2z+1).\end{aligned}\tag{4}$$

Therefore by (3) and (4), we have

$$\frac{\zeta(-2z)}{\Gamma(2z+1)} = -\frac{(2\pi)^{-2z}}{\pi} \sin \pi z \zeta(2z+1).$$

This means  $\tilde{T}(z) = \sin \pi z \tilde{S}(z)$ ,  $\tilde{S} = -\frac{1}{\pi} (2\pi)^{-2z} \zeta(2z+1)$ . Namely,  $T = (\delta_{\pi i} - \delta_{-\pi i}) * S$ , where  $S =$ . Now we replace  $z$  to  $\frac{z-1}{2}$ . Then

$$\begin{aligned}\frac{\zeta(1-z)}{\Gamma(z)} &= -\frac{(2\pi)^{-z+1}}{\pi} \sin \left( \frac{\pi}{2} z - \frac{\pi}{2} \right) \zeta(z) \\ &= 2(2\pi)^{-z} \cos \left( \frac{\pi}{2} z \right) \zeta(z) \\ \zeta(1-z) &= 2(2\pi)^{-z} \cos \left( \frac{\pi}{2} z \right) \Gamma(z) \zeta(z).\end{aligned}\tag{5}$$

Since (5), we obtain “Functional equation of Riemann zeta function”.



## References

- [1] Aomoto-Kita : *Theory of Hypergeometric Functions*, Appendix, (*Selberg Integral and BC type Hypergeometric Functions*), (in Japanese), Springer-Verlag.
- [2] C.A.Berenstein and R.Gay : *Complex Analysis and Special Topics in Harmonic Analysis*, Springer-Verlag (1991).
- [3] R.P.Boas : *Entire Functions*, Academic Press, (1954).
- [4] B.Candelpergher, M.A.Coppo and E.Delabaere : *La sommation de Ramanujan*, L'Enseignement Mathématique, (1997), 93-132.
- [5] F.Dyson : *Statistical theory of the Energy level of complex system I*, J.Math.Phy. Vol.3, (1962), 140-175.
- [6] A.Erdely : *Higher Transcendental Functions Vol.1*, Mcgrawhill, (1953).
- [7] N.N.Khuri : *Regge Pole, Power series*, Physical Review, Vol.132, No.2, (1963), 914-926.
- [8] M.L.Mehta : *Random Matrices*, Academic Press (1990).
- [9] M.Morimoto : *Analytic functionals with non-compact carriers*, Tokyo J.Math. Vol.1, (1978), 77-103.
- [10] M.Morimoto and K.Yoshino : *A uniqueness theorem for holomorphic functions of exponential type*, Hokkaido Math.J. Vol.7, (1978), 259-270.
- [11] H.M.Nussentzweig : *Causality and Dispersion Relation*, Academic Press (1972).
- [12] T.Regge : *Introduction to Complex Orbital Momenta* : Nuovo Cimento, Vol.14, (1959), 951-976.
- [13] E.T.Whittaker and G.N.Watson : *Modern Analysis*, Cambridge University Press, (1927).
- [14] K.Yoshino : *On Carlson's theorem for holomorphic functions*, Algebraic Analysis (Edited by M.Kashiwara and T.Kawai), Vol.2, Academic Press, (1988), 943-950.

- [15] K.Yoshino and M.Suwa : *Plana's summation formula for holomorphic functions of exponential type*, Sûriken Kôkyûroku, No.1158, (2000), 180-189.